Identifiability Conditions for Superior Detection

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***Abstract*—In recent years, various methods are presented for the case of more sources than sensors in the array processing which is called superior detection. Khatri-Rao approach has been proposed for localizing quasi stationary sources. Nested arrays are also used for this purpose. In this Letter, by exploiting the rank of covariance matrix, identifiability of these approaches have been discussed. Conditions for every batch and almost every batch of received signal has been presented.**

***Index Terms*—array processing, direction of arrival, identifiability, superior detection.**

I. Introduction

DIRECTION Of Arrival (DOA) estimation for multiple sources has been an active research area in the past decades [1]. There is a vast use of Uniform Linear Array (ULA) in DOA Estimation. ULA is an array which elements are spaced equally along a straight line. Bresler in [6], shows limitation on the number of signals which ULA can resolve. This fact becomes a motivation towards the development of diverse classes of arrays which are Nonuniform Linear Arrays (NLA). Primal endeavor was by Moffet who proposed Minimum Redundancy Linear Arrays (MRLA) [2], which improves the performance of DOA estimation significantly. Constructing a MRLA requires an exhaustive search through possible combinations. In order to resolve this issue, Pal and Vaidyanathan presented two closed form schemes [3], [4], to extend the effective aperture of arrays. The so called nested array, which is constructed by multiple uniform linear subarrays (ULAs) with different inter-element spacing. The other scheme is formed by two sparse ULAs which satisfies the co-prime relationship for the interelement spacing as well as the number of sensors. Also Ma [5], proposed a technique called Khatri-Rao subspace approach, which is used for quasi-stationary signals. For every technique there is a crucial problem: the identifiability, which is the problem of possibility of unique direction estimation of multiple sources by an array of antennas. This problem is the key issues of many practical considerations.

In 1986 Bresler and Mackovski [6], considered a ULA intercepting narrowband deterministic signals. They showed that the uniqueness problem is equivalent to a linear prediction problem and obtained a condition for a unique solution for any given set of observations and a condition for almost every set of observations. Wax and Ziskind [7], extended the results in [6], for any array configuration. They presented a necessary condition for identifiability of narrowband sources by passive sensor arrays. In 1992, Wax and Ziskind has studied the case wherein the signals are constrained [8].

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The problem for geolocation system is studied in [9], wherein multiple arrays are exploited to estimate the 2'D coordinates of the sources. In our model, sources are assumed to be in far field and therefore, their DOA is the only parameter of interest. Another difference is that, in [9], the Gaussian signal is assumed while in the Khatri-Rao approach investigated in this letter, this is not the case. Abramovich et. al. in [10], investigate the identifiability problem from a detection and estimation point of view. However, our analysis is focused on the estimation part. Because performance of the available methods for source enumeration such as MDL [12], and AIC [13], are investigated independently in the literature [11].

In this letter, we derive conditions for unique localization of multiple sources for Khatri-Rao, and Nested approaches. First, data model of signal is presented. Then, identifiability for every batch and almost every batch of received signal of both approaches are discussed. Furthermore, conditions for unique solution of Nested approach when there is a constraint on signals are presented.

II. Khatri-Rao Subspace Approach

A. Data Model

Assume q narrowband signals impinging on a p-element ULA. Signal DOA set is $\boldsymbol{\theta} = \{\theta_i, i = 1, \ldots, q\}$. The received signal $\boldsymbol{x}(t)$ is a $p \times 1$ vector modeled as:

$$\boldsymbol{x}(t) = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{s}(t) + \boldsymbol{n}(t) \tag{1}$$

$\boldsymbol{s}(t)$, $\boldsymbol{A}(\boldsymbol{\theta})$ are uncorrelated sources and array response (also known as steering matrix), respectively. $\boldsymbol{n}(t)$ is noise component which is assumed to be white and uncorrelated with source signal. In this section, we suppose that source is quasi stationary. It means that source variance is constant in each frame:

$$E[|s_i(t)|^2] = r_i^{(m)} \;\; t \in [(m-1)K, mK-1] \;\; m = 1, \ldots, M \tag{2}$$

Where $s_i(t)$ is i-th source. K and M are the frame length and the number of frames, respectively. Covariance of received signal $\boldsymbol{x}(t)$ in frame m shall be defined as

$$\boldsymbol{R}_x^{(m)} = E[\boldsymbol{x}(t)\boldsymbol{x}(t)^H] \in \mathbb{C}^{p\times p} \quad \forall t \in [(m-1)K, mK-1] \tag{3}$$

Using signal model (1), it becomes:

$$\boldsymbol{R}_x^{(m)} = \boldsymbol{A}\boldsymbol{R}_s^{(m)}\boldsymbol{A}^H + \boldsymbol{R}_n \tag{4}$$

In which $\boldsymbol{R}_s^{(m)}$ is the source covariance matrix in form of

$$\boldsymbol{R}_s^{(m)} = \mathrm{diag}(r_1^{(m)}, r_2^{(m)}, \ldots, r_q^{(m)}) \tag{5}$$

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A local estimation for $\boldsymbol{R}_x^{(m)}$ is :

$$\hat{\boldsymbol{R}}_x^{(m)} = \frac{1}{K} \sum_{t=(m-1)K}^{mK-1} \boldsymbol{x}(t)\boldsymbol{x}(t)^H \tag{6}$$

For DOA estimation in this situation, a well-known approach called Khatri-Rao Subspace is used [5]. In this method, for each frame, signal covariance of received signal is vectorized.

$$\boldsymbol{y}_m \triangleq vec(\boldsymbol{R}_x^{(m)}) = \text{vec}(\boldsymbol{A}\boldsymbol{R}_s^{(m)}\boldsymbol{A}^H) + \text{vec}(\boldsymbol{R}_n) \tag{7}$$

Since $\boldsymbol{R}_s^{(m)}$ is diagonal, Khatri-Rao product can be used.

$$\boldsymbol{y}_m = (\boldsymbol{A}^* \otimes \boldsymbol{A})\boldsymbol{r}_m + \text{vec}(\boldsymbol{R}_n) \tag{8}$$

Where $(\cdot)^*$ and $\otimes$ stand for complex conjugate and Khatri-Rao product, respectively. $\boldsymbol{r}_m = [r_1^{(m)}, r_2^{(m)}, \ldots, r_q^{(m)}]^T$. Then, $\boldsymbol{Y}$ is constructed by collecting $\boldsymbol{y}_m$ of each frame as its columns.

$$\boldsymbol{Y} = [\boldsymbol{y}_1, \ldots, \boldsymbol{y}_M] \tag{9}$$

$$\boldsymbol{Y} = (\boldsymbol{A}^* \otimes \boldsymbol{A})\boldsymbol{\psi} + \text{vec}(\boldsymbol{R}_n)\mathbf{1}_M^T \tag{10}$$

$$\boldsymbol{\psi} = [\boldsymbol{r}_1, \ldots, \boldsymbol{r}_M] = \begin{bmatrix} r_{11} & r_{12} & \ldots & r_{1M} \\ r_{21} & r_{22} & \ldots & r_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ r_{q1} & r_{q2} & \ldots & r_{qM} \end{bmatrix} \tag{11}$$

Where $\mathbf{1}_M^T$ is a row matrix of M ones. Then, noise component will be omitted by a projection:

$$\boldsymbol{P}^\perp = \boldsymbol{I}_M - \frac{1}{M}\mathbf{1}_M\mathbf{1}_M^T \tag{12}$$

At last, using Singular Value Decomposition (SVD) noise subspace can be used in subspace algorithms like MUSIC [1].

$$\boldsymbol{Y}\boldsymbol{P}^\perp = [\boldsymbol{U}_s \quad \boldsymbol{U}_n] \begin{bmatrix} \boldsymbol{\Sigma}_s & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{V}_s^H \\ \boldsymbol{V}_n^H \end{bmatrix} \tag{13}$$

B. Identifiability For Khatri-Rao Approach

1) Every Batch: Now condition is explained under which the solution will be identifiable for every batch Y. As projection has no effect on our analyses, it is ignored. $\boldsymbol{B} = \boldsymbol{A}^* \otimes \boldsymbol{A}$. For supposed received signals, the DOA will be unique when:

$$\boldsymbol{Y} = \boldsymbol{B}(\boldsymbol{\theta})\boldsymbol{\psi}^T \neq \boldsymbol{B}(\boldsymbol{\theta}')\boldsymbol{\psi}^T \tag{14}$$

$$[\boldsymbol{B}(\boldsymbol{\theta}) \quad \boldsymbol{B}(\boldsymbol{\theta}')] \begin{bmatrix} \boldsymbol{\psi}^T \\ \boldsymbol{\psi}'^T \end{bmatrix} \neq 0 \tag{15}$$

If (15) holds, the nullity of $[\boldsymbol{B}(\boldsymbol{\theta}) \quad B(\boldsymbol{\theta}')]$ which is assumed to be ζ must be smaller than $V = \text{Rank} \begin{bmatrix} \boldsymbol{\psi}^T \\ \boldsymbol{\psi}'^T \end{bmatrix}$. Since

$$V \geq \eta = \text{Rank}[\boldsymbol{\psi}^T] \tag{16}$$

It's enough that $\zeta < \eta$.

Two mutually exclusive cases for d are considered, depending on the number of repetitions of equal directions in two sets of solutions: $d = \#\text{pair}\{\theta_i = \theta_j\}$.

Case 1: $d < 2q - (2p-1)$. Since $\zeta = 2q - \text{Rank}[\boldsymbol{B}(\boldsymbol{\theta}) \quad \boldsymbol{B}(\boldsymbol{\theta}')]$ and $\text{Rank}[\boldsymbol{B}(\boldsymbol{\theta})]$ is equal to minimum of its dimensions $\{q, 2p-1\}$. Hence, $\zeta = 2q - \min\{2p-1, 2q-d\}$. Therefore, for $\zeta < \eta$, we must have:

$$q < \frac{\eta + 2p - 1}{2} \tag{17}$$

Case 2: $d > 2q - (2p-1)$. First we omit those redundant column where $\theta_i = \theta_j$ and call it $\overline{\boldsymbol{B}}$

$$\zeta = (2q - d) - \text{Rank}[\overline{\boldsymbol{B}}(\boldsymbol{\theta}) \quad \overline{\boldsymbol{B}}(\boldsymbol{\theta}')] = 0 \tag{18}$$

Since $V > 0$, therefore, $\zeta < V$ is always true.

2) Almost Every Batch: Now, we discuss almost every batch $\boldsymbol{Y}$ with exception of a set of batches of measure zero. Let the set of all legitimate matrices $\boldsymbol{Y}$ denoted by $\boldsymbol{Y}(\boldsymbol{\theta}) = \{\boldsymbol{Y} | \boldsymbol{Y} = \boldsymbol{B}(\boldsymbol{\theta})\boldsymbol{\psi}^T\}$. We assume that $\boldsymbol{\theta}$ and $\boldsymbol{\theta}'$ are two ambiguous solution sets, we define $D(\boldsymbol{\theta}) = \bigcup_{\boldsymbol{\theta}'} D(\boldsymbol{\theta}, \boldsymbol{\theta}')$ where $D(\boldsymbol{\theta}, \boldsymbol{\theta}') = \{\boldsymbol{Y} | \boldsymbol{Y} = \boldsymbol{B}(\boldsymbol{\theta})\boldsymbol{\psi}^T = \boldsymbol{B}(\boldsymbol{\theta}')\boldsymbol{\psi}'^T\}$). Since $\text{Rank}(\boldsymbol{Y}) = \mu$ and every column of it is in the span of $\{\boldsymbol{B}(\boldsymbol{\theta})\}$ of rank $2p-1$. Therefore, $(2p-1)\mu$ complex parameters are required to describe $\boldsymbol{Y}$.

$$\dim \boldsymbol{Y}(\boldsymbol{\theta}) = 2(2p-1)\eta \tag{19}$$

$$\dim D(\boldsymbol{\theta}, \boldsymbol{\theta}') = 2\eta\zeta \tag{20}$$

Now we assume two cases for d. Case 1: $d < 2q - (2p-1)$. As $\zeta = 2q - (2p-1)$, dimension of $D(\boldsymbol{\theta}, \boldsymbol{\theta}')$ will be equal to $2(2q-(2p-1))\eta$. Since dim $(\boldsymbol{\theta}') = q$, dimension of $D(\boldsymbol{\theta})$ will be less than or equal to $2(2q-(2p-1))\eta + q$. For dim $D(\boldsymbol{\theta}) <$ dim $\boldsymbol{Y}(\boldsymbol{\theta})$ we must have:

$$q < \frac{4\eta}{4\eta + 1}(2p-1) \tag{21}$$

Case 2: $d > 2q - (2p-1)$. In this case dimension of $D(\boldsymbol{\theta}, \boldsymbol{\theta}')$ will be $2d\eta$. Therefore, dimension of $D(\boldsymbol{\theta})$ will be less than or equal to $2d\eta + q$. For dim $D(\boldsymbol{\theta}) <$ dim $\boldsymbol{Y}(\boldsymbol{\theta})$ we must have:

$$q < \frac{2\eta}{2\eta + 1}(2p-1) \tag{22}$$

Since $\eta > 1$, the last inequality is more tight.

3) Constrained Signals: In this part, we consider real constraint on signals.

$$f_k(E|s_k(t_i)|^2, \lambda_k^\mu) = 0 \quad k = 1, \ldots, q \tag{23}$$

Where $E|s_k(t_i)|^2$ is second order statistics of kth source at time t_i, f_k is a smooth map from the complex plane to real line. λ_k^μ is a vector of parameters with μ real unknowns. In fact, this constraint describes the relation of second order statistics of the sources by a function.

The number of free parameters to describe $\boldsymbol{B}(\boldsymbol{\theta})$ is $2p-1$. The source equivalent $\boldsymbol{\psi}^T$ is $q \times M$ real matrix with η-column rank, which is described by $q\eta$ free real parameters. Let the set of all $\boldsymbol{\psi}$, rank-η matrix which comply with above constraint, be denoted by D. The number of free parameters is reduced by constraint to $q\eta - q\mu$. Therefore, dimension of D is $q\eta - (q\eta - q\mu) = q\mu$. Set of ambiguous solution is similar to "almost every batch" section defined $D(\theta) = \bigcup_{\theta'}$.

For identifiability, its enough that the dimension of the set $D(\theta)$ be smaller than dimension of D. Similar to the last section, in two cases we have:

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Case 1: $d < 2q-(2p-1)$. As $\zeta = \text{null}[\boldsymbol{B}(\boldsymbol{\theta}) \quad \boldsymbol{B}(\boldsymbol{\theta}')]$, and $\text{Rank}[\boldsymbol{B}(\boldsymbol{\theta})\boldsymbol{B}(\boldsymbol{\theta}')] = \min\{2p-1, 2q-d\}$. Hence, $\zeta = 2q-(2p-1)$. Thus, $2(2q-(2p-1))$ real parameters are required to describe an arbitrary vector in null space of $[\boldsymbol{B}(\boldsymbol{\theta}) \quad \boldsymbol{B}(\boldsymbol{\theta}')]$ and $2(2q-(2p-1))\eta$ are required to describe an arbitrary rank-η matrix $\begin{bmatrix}\boldsymbol{\psi}^T \\ \boldsymbol{\psi}'^T\end{bmatrix}$ in this null space. Therefore, dim $D(\boldsymbol{\theta}, \boldsymbol{\theta}') = 2(2q-(2p-1))\eta-(q\eta-q\mu)$. Since dimension of $D(\theta')$ is q, Therefore

$$\text{dim } D(\boldsymbol{\theta}, \boldsymbol{\theta}') = 2(2q-(2p-1))\eta - (q\eta - q\mu) \tag{24}$$

For dim $D(\boldsymbol{\theta}) <$ dim D we must have :

$$q < \frac{2\eta}{3\eta+1}(2p-1) \tag{25}$$

As we can see, number of unknown parameters of constraint have the same effect on dimension of d and dim $D(\boldsymbol{\theta})$. While it does not affect the condition.

Case 2: $d > 2q-(2p-1)$. According to the last section, we omit the redundant columns. Therefore

$$\zeta = (2q-d) - \text{Rank}[\overline{\boldsymbol{B}}(\boldsymbol{\theta}) \quad \overline{\boldsymbol{B}}(\boldsymbol{\theta}')] = 0 \tag{26}$$

and dim $D(\boldsymbol{\theta}, \boldsymbol{\theta}') = 0-(q\eta-q\mu)$. Therefore, dim $D(\boldsymbol{\theta}) < -(q\eta-q\mu)+q$. If $\eta > 1$ then dim $D(\boldsymbol{\theta}) <$ dim D.

4) Alternative Proof For Constrained Signals: $\boldsymbol{\psi}^T$ is described by $q\eta$ free real parameters. The number of free parameters is reduced by constraint and becomes $(q\eta - q\mu)$. Since $\boldsymbol{\psi}$ is affected by q source, the dimensionality of legitimate set is equal to $q\mu + q$ ambiguous solution:

$$C \triangleq \boldsymbol{B}(\boldsymbol{\theta})\boldsymbol{\psi}^T \neq B(\boldsymbol{\theta}')\boldsymbol{\psi}^T \tag{27}$$

The ambiguity set is defined by

$$D = \{\boldsymbol{B}(\boldsymbol{\theta})\boldsymbol{\psi}^T : C = 0 \quad \text{constrained}\} \tag{28}$$

$\boldsymbol{B}(\boldsymbol{\theta})\boldsymbol{\psi}^T$ has $2q\eta + q$ complex parameters. The constraint reduces it by $(2p-1)\eta$. Therefore, its dimension will be $2(2q\eta-(2p-1)\eta+q)$. For unique solution, the dimensionality of the ambiguity set should be smaller than dimensionality of legitimate set, which means $2(2q\eta-(2p-1)\eta+q) < q\eta+q$ which results in (25).

III. NESTED APPROACH

A. Data Model

Two-level nested array consists of two ULA, in which the inner one has L_1 elements with spacing d_1, and the outer has L_2 with spacing $d_2 = (L_1+1)d_1$. Therefore, the location of each element is

$$\begin{aligned} d = [&d_1, 2d_1, \ldots, L_1d_1, (L_1+1)d_1, \\ &2(L_1+1)d_1, \ldots, L_2(L_1+1)d_1] \end{aligned} \tag{29}$$

Define

$$D = \{d_i - d_j | i, j = 1, \ldots, L = L_1 + L_2\} \tag{30}$$

This array can act as a virtual array with this element:

$$d_v = [-kd_1, \ldots, kd_1] \quad k = L_2(L_1+1)-1 \tag{31}$$

In the nested approach, covariance matrix of received signal is

$$\boldsymbol{R}_x = \boldsymbol{A}\boldsymbol{R}_s\boldsymbol{A}^H + \sigma^2\boldsymbol{I}_L \tag{32}$$

Where σ^2 is the noise variance and $\boldsymbol{R}_S = \text{diag}(\sigma_1^2, \ldots, \sigma_q^2)$ is covariance of uncorrelated sources. Now, z is defined as:

$$\boldsymbol{z} \triangleq \text{vec}(\boldsymbol{R}_x) \tag{33}$$

Consequently

$$\boldsymbol{z} = (\boldsymbol{A}^* \otimes \boldsymbol{A}^H)\boldsymbol{p} + \sigma^2 1_L \tag{34}$$

In which $\boldsymbol{p} = [\sigma_1^2, \ldots, \sigma_q^2]^T$, and $1_L = [\boldsymbol{e}_1^T, \ldots, \boldsymbol{e}_L^T]^T$ where $\boldsymbol{e}_i$ is a vector with the i-th element of 1 and the others of 0. Construct $\boldsymbol{A}_1$ by omitting redundant rows of $\boldsymbol{A}^* \otimes \boldsymbol{A}^H$, Therefore

$$\boldsymbol{z}_1 = \boldsymbol{A}_1\boldsymbol{p} + \sigma^2\boldsymbol{e}' \tag{35}$$

Where $\boldsymbol{e}'$ is a $(L^2+2L-2)/2 \times q$ vector in which the $L^2/4+L/2$-th element is 1 and the others are 0.

B. Identifiability For Nested Approach

1) Every Batch: The condition will be derived under which the solution will be unique for every batch of data.

$$\boldsymbol{z}_1 = \boldsymbol{A}_1(\boldsymbol{\theta})\boldsymbol{p} \neq \boldsymbol{A}_1'(\boldsymbol{\theta}')\boldsymbol{p}' \tag{36}$$

$$[\boldsymbol{A}_1(\boldsymbol{\theta}) \quad \boldsymbol{A}_1'(\boldsymbol{\theta}')]\begin{bmatrix}\boldsymbol{p} \\ \boldsymbol{p}'\end{bmatrix} \neq 0 \tag{37}$$

If (37) holds, nullity of $[\boldsymbol{A}_1(\boldsymbol{\theta})\boldsymbol{A}_1'(\boldsymbol{\theta}')]$ $(= \zeta)$ must be smaller than $V = \text{Rank}\begin{bmatrix}\boldsymbol{p} \\ \boldsymbol{p}'\end{bmatrix}$. Since $V > \eta = \text{Rank}[\boldsymbol{p}]$, It is enough that $\zeta < \eta$ also in this situation, we have two mutual exclusive cases for d. For case $d < 2q-(L^2+2L-2)/2$:

$$q < \frac{\eta + (L^2+2L-2)/2}{2} \tag{38}$$

For the complement case, it can be shown that, there is no restrictive condition.

2) Almost Every Batch: It is straightforward to show:

$$q < \frac{2\eta}{2\eta+1}(L^2+2L-2)/2) \tag{39}$$

3) Constrained Signals:

$$q < \frac{2\eta}{3\eta+1}(L^2+2L-2)/2) \tag{40}$$

IV. CONCLUSION

In this letter, conditions for identifiability of signals in both Khatri-Rao and Nested approaches for every batch, almost every batch, and the constrained data were derived. It is proved that the number of resolvable sources is $\mathcal{O}(L^2)$ and $\mathcal{O}(2p)$ for Nested and Khatri-Rao approaches, respectively. The gap between the performance of the existing methods and the upper bounds in this letter can be closed by development of better algorithms for superior detection in array signal processing.

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